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## Modeling and Control of Systems with Active Singularities Under Energy Constraints: Single- and Multi-Impact Sequences

Joseph Bentsman, Boris M. Miller, Evgeny Ya. Rubinovich, and Sudip K. Mazumder

**Abstract**—A controller synthesis setting for systems with controlled singularities under incomplete information is extended to the cases characterized by a constraint on the total physical impulse as well as on the time interval between the adjacent engagement phases. The use of this setting for the optimal control law calculation is illustrated on the ball/racket system representative of a large class of controlled impact problems, where there are two separate bodies interacting through controlled viscoelastic force.

**Index Terms**—Control over observations, controlled singularities, energy constraints, impulse control, mechanical systems.

### I. INTRODUCTION

Dynamical systems with active, or controlled, singularities is a new class of systems [1] characterized by the presence of *active, or controlled, constraints* capable of radically changing the attainability set of the post-impact system state. The engagement phase of the system with such constraint is termed *active singularity*, and the system motion in the domain of constraint violation—the *singular* motion phase. Based on these concepts, the corresponding rigorous technique for modeling and controller synthesis for systems with active singularities of elastic

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J. Bentsman is with Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: jbentsma@illinois.edu).

B. M. Miller is with the School of Mathematical Sciences, Monash University, Clayton, 3800, Vic., Australia and also with the Institute for Information Transmission Problems, Moscow, Russia (e-mail: boris.miller@sci.monash.edu.au).

E. Y. Rubinovich is with the Institute of Control Sciences, Moscow 117997, Russia (e-mail: rubinvch@ipu.rssi.ru).

S. K. Mazumder is with the Department of Electrical and Computer Engineering, University of Illinois at Chicago, Chicago, IL 60607 USA (e-mail: mazumder@ece.uic.edu).

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type under full state accessibility was developed in [1]. [2] extended the setting of [1] to admit an incomplete observation in the singular motion phase. To accommodate the output feedback optimal controller synthesis in the singular phase, [2] introduced the novel dynamic mode of system interaction with the constraints, termed *temporal multi-impact*, characterized by the very short duration isolated set of temporally sequenced control signals. This mode was shown to vastly increase attainability set of the post-temporal-multi-impact system state in comparison to the single-impact mode considered in [1]. To adequately represent temporal multi-impact, [2] also introduced a new mode of system behavior—the *interlaced singular phase*—an engagement phase exhibited by the system under a temporal multi-impact, and generalized the framework of [1] to encompass this mode. [2] also presented an extensive example of calculation of the single temporal multi-impact control laws for the open loop and the observation-based optimal ball stopping in an elastic ball/racket system.

Practical realization of controlled singularities [1], [2] in a number of systems is, however, characterized by a constraint on the total physical impulse that accounts for the energy depletion during an engagement phase, as well as on the time interval between adjacent engagement phases to secure the energy recovery. The optimization problem that arises in this case is fundamentally different from the magnitude constraint one considered in [2]. Integrating the material given in [3] and [4] into a succinct presentation, this paper extends the framework of [1] and [2] to permit the design of the observations-based optimal control laws under energy constraints. The latter are shown on a rather non-trivial ball/racket system example to require the single-impact and/or the temporal multi-impact finite control sequences to attain the desired control objective. The applications that stand to benefit from controller design for this case include, among others, power systems under faults and ultra-fast microgrids [8], mobile sensor networks, impact actuators [10], and robotic manipulators [5]–[7], [9].

### II. GENERAL PROBLEM STATEMENT

**Motion in the Regular and the Singular Phases:** Let the controlled dynamical system be described by the state vector  $x(t) = (x_p(t), x_v(t))$ ,  $x_p(t) \in R^n$ ,  $x_v(t) \in R^n$ , where vectors  $x_p$  and  $x_v$  are referred to as the sets of *generalized positions* and *generalized velocities*, respectively, and  $t \in [0, T]$ , where  $T$  is sufficiently large.

Suppose that system motion includes interaction with constraint that undergoes the elastic deformation parametrized by some coefficient  $\mu > 0$ , so that for finite  $\mu$  the constraint would admit a system motion, although inhibited, within the domain occupied by it. Let the constraint-free domain be given by  $\{(x_p, t) : G(x_p, t) > 0\}$  where  $G : R^n \times [0, T] \rightarrow R$  is a sufficiently smooth function. Following [1], the system motions in the domain occupied by the constraint and in the constraint-free domain will be referred to as the *singular* and the *constraint-free* motion phases, respectively.

Let the system motion be described by a system

$$\begin{aligned} \dot{x}_p(t) &= F_p^r(x_p(t), x_v(t), t), \dot{x}_v(t) = F_v^r(x_p(t), x_v(t), u(t), t) \\ &+ \mu F_v^s(x_p(t), x_v(t), w_1^\mu(\xi, t), t, \mu) I\{t : G(x_p(t), t) \leq 0\} \\ &+ \mu F_v^{rs}(x_p(t), x_v(t), w_2^\mu(\xi, t), t, \mu) \end{aligned} \quad (1)$$

where  $u(t) \in U \subset R^r$  is a generalized control in the constraint-free phase that is unbounded in  $R^r$ , but restricted in the integral sense:  $\int_0^T |u(t)| dt \leq M < \infty$ ,  $M = \text{const}$ ,  $F_p^r(x_p, x_v, u, t)$  and  $F_v^r(x_p, x_v, u, t)$  are the generalized forces in the constraint-free phase,  $w_1^\mu(\xi, t)$  is an external control signal in the singular phase,  $\mu F_v^s(x_p, x_v, w_1^\mu(\xi, t), t, \mu)$  is a generalized controlled force in the

singular phase that includes the force arising from a contact with the constraint in the inhibited area as well as from the external impulse,  $\mu F_v^{rs}(x_p, x_v, w_2^\mu(\xi, t), t, \mu)$  is an additional generalized controlled force (impulse action) in the constraint-free phase governed by a control signal  $w_2^\mu(\xi, t)$  (a measurable function), and  $\xi$  is the sensor output signal, where  $I\{t : G(x_p(t), t) \leq 0\}$  is the indicator function of the appropriate set.

The first of the latter two forces,  $\mu F_v^s(x_p, x_v, w_1^\mu(\xi, t), t, \mu)$ , is characterized by

$$\begin{aligned} F_v^s(x_p, x_v, w_1^\mu, t, \mu) &= 0, \text{ if } 1) G(x_p, t) > 0 \\ &\text{or } 2) G(x_p, t) = 0 \text{ and} \\ &\left. \frac{d}{dt} \right|_{F_p^r} G(x_p, t) = G'_{x_p}(x_p, t) F_p^r(x_p, x_v, t) \\ &\quad + G'_t(x_p, t) = 0 \end{aligned} \quad (2)$$

where  $G'_{x_p}$  and  $G'_t$  denote partial derivatives with respect to  $x_p$  and  $t$ , respectively, and  $(d/dt)|_{F_p^r} G(x_p, t)$  denotes the time derivative of  $G(x_p, t)$  along the trajectories of  $\dot{x}_p(t) = F_p^r(x_p(t), x_v(t), t)$ . Noting that  $G(x_p, t)$  does not depend on  $x_v$ , the last expression in (2) is seen to represent the time derivative of  $G(x_p, t)$  along the trajectories of the entire system (1).

The force  $\mu F_v^{rs}(x_p, x_v, w_2^\mu(\xi, t), t, \mu)$ , the last of the forces in (1), characterizes an external impulsive action on the system in the constrained-free domain during the so-called inter-singular motion formally defined in [2], p. 1748, and satisfies the condition  $F_v^{rs}(x_p, x_v, w_2^\mu, t, \mu) = 0$ , if  $G(x_p, t) < 0$ . The introduction of this force lays the groundwork for addressing optimal control problems with complex temporal multi-impact structure. Whether or not the temporal multi-impact will appear depends on the specific features of the problem at hand.

*Sensor Equations, Constraints, and General Problem Statement:* Let in the singular phase, when  $G(x_p(t), t) \leq 0$ , components of the state vector  $(x_p(t), x_v(t))$  be unobservable directly, and it be possible to observe only the sensor output signal  $\xi(t) \in R^k$  that is assumed to satisfy the equation

$$\xi(t) = \mu H(x_p(t), x_v(t), t, \mu) \quad (3)$$

where  $H(x_p, x_v, t, \mu) = 0$  if  $G(x_p, t) > 0$ . Then, the control signals  $w_1^\mu(\xi, t)$  and  $w_2^\mu(\xi, t)$  can be taken to be continuous functionals of the sensor output signal  $\xi(t)$  and measurable in time (cf. Definition 1 of [2]). Let the motion in the singular phase begin at  $\tau$ , where  $\tau$  is the first instant when

$$G(x_p(\tau), \tau) = 0 \quad \text{and} \quad \left. \frac{d}{dt} \right|_{F_p^r} G(x_p(\tau), \tau) < 0. \quad (4)$$

Denoting by  $\gamma$  any of the control signals  $w_1, w_2$ , define its dependence on  $t$  and  $\mu$  in the singular (interlaced singular) phase as

$$\gamma^\mu(\xi, t) = \begin{cases} \gamma(\xi, \sqrt{\mu}(t - \tau)), & t \geq \tau, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

It is assumed that the right-hand sides of (1)–(3) are sufficiently smooth to guarantee unique solution of (1)–(3) for any admissible controls.

Assume that force  $F_v^r(x_p(t), x_v(t), u(t), t)$  in the regular phase takes the form of an impulsive force  $F_v^r(t_k)\delta(t - t_k)$ , satisfying

$$|F_v^r(t_k)| \leq F_0 \quad (6)$$

where  $F_0$  is given. The instants  $t_k, k = 1, 2, \dots$ , of the impulse actions are assumed to satisfy the monotonicity condition  $\tau = t_0 < t_1 < t_2 < \dots$  and the explicit restrictions on the impulse repetition rate of the form

$$T_k \leq t_k - t_{k-1} \quad (7)$$

where  $T_k$  are the given nonnegative constants that represent limitations of control capability in the regular phase.

Let  $F(w_1^\mu(\xi, t), \mu)$  be part of  $\mu F_v^s(x_p, x_v, w_1^\mu(\xi, t), t, \mu)$  representing the external force acting in a singular phase. In contrast to [1] and [2], let physically motivated restrictions be specified as the *constraint on the total external impulse*

$$\int_{\tau_i^\mu}^{\tau_i^{\mu*}} |F(w_1^\mu(\xi, t), \mu)| dt \leq C_i \quad (8)$$

acting in an  $i$ -th singular phase, and constraint

$$T_k \leq t_k - \tau_i^{\mu*} \quad (9)$$

on the time interval between the end of the singular phase and the next impulse in the regular phase. Here,  $C_i$  are given constants and  $[\tau_i^\mu, \tau_i^{\mu*}]$  is an  $i$ -th singular phase time interval.

The present work addresses two *general objectives*: 1) *controller synthesis setting objective*: provide an analytical setting that permits reduction of an ill-posed problem of synthesis of the singular phase control signals  $w_1^\mu(\xi, t)$  and  $w_2^\mu(\xi, t)$  in (1)–(3) under constraints (6)–(9) to a well-posed two-step approximation procedure: a) synthesis of bounded singular phase control signals  $w_1(\eta, s)$  and  $w_2(\eta, s)$  in the auxiliary fictitious time  $s$  with sensor data  $\eta(s)$  under these constraints, and b) calculation of  $w_1^\mu(\xi, t)$  and  $w_2^\mu(\xi, t)$  implementable in the original system (1)–(3) using signals synthesized in a); 2) *limit modeling objective*: obtain a model that generates a discontinuous motion controlled by  $w_1(\eta, \cdot)$  and  $w_2(\eta, \cdot)$  representing a consistent approximation of motion of (1) controlled by  $w_1^\mu(\xi, t)$  and  $w_2^\mu(\xi, t)$ . Tasks a) and b) of 1) can be viewed as the direct and the converse ones, respectively. These objectives for a single impact are addressed by Theorem 1 and for a temporal multi-impact sequence—by Theorem 2.

### III. EXAMPLE

Assume that the racket of mass  $M$  and the ball are moving under zero gravity along the horizontal axis with constant speeds for  $t < \tau$ , where  $\tau$  is the first impact time. The lack of gravity assumption only trivially alters the system behavior.

This system has the state vector  $X = (x_p, x_v, X_p, X_v)$ , where  $x_p, x_v$  and  $X_p, X_v$  are the positions and the velocities of the ball and the racket, respectively. The area free of constraint is described by the inequality  $G(X) = x_p - X_p > 0$ . In the example considered in [2] the regular motion phase control signal  $u(t)$  was not invoked. In contrast to this, in the present example it is assumed that during the regular phase the racket can be subject to an impulsive force  $F^r(t_k)$ , satisfying (6). The instants  $t_k, k = 1, 2, \dots$ , of the impulse actions are assumed to satisfy the monotonicity condition  $\tau = t_0 < t_1 < t_2 < \dots$  and the restriction (7). In the regular phase the equations of motion have the form

$$\begin{aligned} \dot{x}_p(t) &= x_v(t), & \dot{X}_p(t) &= X_v(t) \\ \dot{x}_v(t) &= 0, & \dot{X}_v(t) &= M^{-1} \sum_{t_k \leq t} F^r(t_k)\delta(t - t_k). \end{aligned} \quad (10)$$

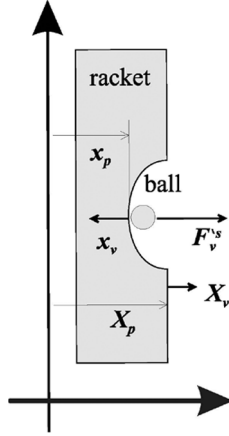


Fig. 1. Singular motion phase: motion in the inhibited domain.

Here  $\delta(\cdot)$  is a Dirac  $\delta$ -function.

In the inhibited area, depicted in Fig. 1, the motion is described by the equations

$$\begin{aligned} \dot{x}_p(t) &= x_v(t), \quad \dot{x}_v(t) = -\mu F_v^{ls}(X(t), \mu), \quad \dot{X}_p(t) = X_v(t) \\ \dot{X}_v(t) &= M^{-1} [F(t, \mu) + \mu F_v^{rs}(X(t), \mu)]. \end{aligned} \quad (11)$$

In (11)  $\mu F_v^{ls}(X(t), \mu)$  is a viscoelastic force during the contact of the ball and the racket. This force, described by  $F_v^{ls}(X(t), \mu) = x_p(t) - X_p(t) + 2\kappa\mu^{-1/2}(x_v(t) - X_v(t))$ , appears in both the ball and the racket velocity equations with the opposite signs. The force  $F(t, \mu)$  could be interpreted as an external control force acting on the racket in the singular phase. This force admits the representation

$$F(t, \mu) = \sqrt{\mu} w(\sqrt{\mu}(t - \tau)), \quad t \geq \tau. \quad (12)$$

Here  $\mu > 0$  is the elasticity coefficient,  $0 \leq \kappa \leq 1$  is the damping, and  $w(\cdot)$  is a control variable satisfying the constraint

$$|w(\cdot)| \leq w_0 < \infty. \quad (13)$$

In contrast to [2], there is the constraint on total external impulse (8), acting on the racket in an  $i$ -th singular phase, and constraint (9) on the time interval between the end of the singular phase and the next impulse in the regular phase.

Suppose a player (or robot) has the capability of measuring the pressure on the racket during the contact phase. This implies that the sensor output signal  $\xi(t)$  is equal to the viscoelastic force acting on the racket

$$\xi(t) = \mu [x_p(t) - X_p(t) + 2\mu^{-1/2}\kappa(x_v(t) - X_v(t))]. \quad (14)$$

Equations (10)–(14) describe the motion in the case of  $\mu < \infty$ .

System (11), (12) is easily recast into (1) by defining  $x_p$  and  $x_v$  of (1) as two-vectors  $[x_{p1}, x_{p2}]^T$  and  $[x_{v1}, x_{v2}]^T$  given by  $[x_p, X_p]^T$  and  $[x_v, X_v]^T$ , respectively. The functions  $F_p^r, F_v^r, \mu F_v^s$ , and  $\mu F_v^{rs}$  in the rhs of (1) are then simply given in terms of functions  $\mu F_v^{ls}, F$ , and  $F^{rs}$  defined in (11), (12), as for example, shown by expressions (29)–(31), p. 1747, in [2].

As indicated in above subsection in this case  $F_v^{rs}(x_p, x_v, w_2^u(\xi, t), t, \mu) \equiv 0$ , so that control  $w_2$  in (5) is absent.

*Infinitesimal Dynamics Equation Under a Single Impact:* According to (4), the singular motion phase begins at the first time  $\tau$  that the system engages the constraint. Therefore, for a finite value of  $\mu$  there exists a non-zero time interval of the constraint violation. Theorem given next provides analytical setting for optimal controller synthesis under constraints.

*Assumption 1:* Suppose that  $F_v^s$  (analogously  $F_v^{rs}$  and  $H$ ) satisfies the Lipschitz condition in the following form: there exist  $L > 0, \mu_0 > 0$  such that for any  $(x_p, x_p', x_v, x_v'), t \in [0, T], w_1 \in W_1$ , and  $\mu \geq \mu_0, \|F_v^s(x_p, x_v, w_1, t, \mu) - F_v^s(x_p', x_v', w_1, t, \mu)\| \leq L\{\|x_p - x_p'\| + \mu^{-1/2}\|x_v - x_v'\|\}$ .

*Theorem 1:* Along with Assumption 1, let  $(x_p^\mu(t), x_v^\mu(t))$  denote the ordinary solution of the original system (1) where a superscript  $\mu$  is used to indicate dependence of this solution on parameter  $\mu$ . Denote  $t^\mu(s) = \tau + \mu^{-1/2}s$  and assume that:

- 1) for any admissible controls  $w_1$  and for any  $(x_p, \tau)$  such that  $G(x_p, \tau) = 0$  and  $(d/dt)|_{F_p^r} G(x_p(\tau), \tau) < 0$  there exists

$$\begin{aligned} \lim_{\mu \uparrow \infty} \sqrt{\mu} F_v^s \left( \frac{y_p - x_p}{\sqrt{\mu}} + x_p, y_v, w_1(\eta^\mu, s), t^\mu(s), \mu \right) &= \bar{F}_v^s(y_p, y_v, w_1(\eta, s), x_p, \tau) \\ \lim_{\mu \uparrow \infty} \sqrt{\mu} H \left( \frac{y_p - x_p}{\sqrt{\mu}} + x_p, y_v, t^\mu(s), \mu \right) &= \bar{H}(y_v, x_p, \tau) \end{aligned} \quad (15)$$

where convergence is uniform in any bounded vicinity of  $(y_p, y_v, \eta, s)$ ;

- 2) for the system of differential equations

$$\begin{aligned} \dot{y}_p(s) &= F_p^r(x_p(\tau), y_v(s), \tau) \\ \dot{y}_v(s) &= \bar{F}_v^s(y_p(s), y_v(s), w_1(\eta, s), x_p(\tau), \tau) \\ \eta(s) &= \bar{H}(y_v(s), x_p(\tau), \tau) \end{aligned} \quad (16)$$

with  $y_p(0) = x_p(\tau), y_v(0) = x_v(\tau-), \eta(0) = \xi(\tau)$  there exists  $s^*(\tau)$ , such that  $s^*(\tau) = \inf_{s \in A} s$

$$A \triangleq \left\{ s > 0 : \begin{array}{l} G'_t \Big|_{(x_p(\tau), \tau)} s + G'_{x_p} \Big|_{(x_p(\tau), \tau)} \\ \times (y_p(s) - x_p(\tau)) = 0 \\ G'_t \Big|_{(x_p(\tau), \tau)} + G'_{x_p} \Big|_{(x_p(\tau), \tau)} \\ \times F_p^r(x_p(\tau), y_v(s), \tau) > 0 \end{array} \right\} \quad (17)$$

and the systems has the unique solution on some interval  $[0, s^*(\tau) + \varepsilon]$ , where  $\varepsilon > 0$ .

Also define  $\Psi_v(\cdot)$  as a  $v$ -component of the shift operator along the paths of (16) (cf. (28)–(32) of [1]) so that

$$y_v(s^*(\tau)) = y_v(0) + \Psi_v(y_p(0), y_v(0), w_{1\tau}(\cdot), \tau)$$

where  $w_{1\tau}(\cdot) = \{w_1(\eta, s) : 0 \leq s \leq s^*(\tau)\}$ .

Then

- i) if  $\mu \rightarrow \infty$

$$(y_p^\mu(s), y_v^\mu(s), \eta^\mu(s)) \rightarrow (y_p(s), y_v(s), \eta(s)) \quad (18)$$

uniformly on  $[0, s^*(\tau) + \varepsilon]$ , and for all sufficiently large  $\mu$  there exists  $s_\mu^*(\tau) = \inf_{s \in A_\mu} s$

$$A_\mu \triangleq \left\{ s > 0 : \frac{d}{ds} \Big|_{F_p^r} G(x_p(t^\mu(s)), t^\mu(s)) = 0 \right\} \quad (19)$$

such that

$$s_\mu^*(\tau) \rightarrow s^*(\tau); \quad (20)$$

- ii) the generalized solution  $(\bar{x}_p(t), \bar{x}_v(t), \bar{\xi}(t))$  of the original system (1), (3) is a pointwise limit of its ordinary solution as

$\mu \rightarrow \infty$ , and satisfies on an interval  $[0, \tau + \varepsilon]$  the system of generalized differential equations

$$\begin{aligned} \dot{\bar{x}}_p(t) &= F_p^r(\bar{x}_p(t), \bar{x}_v(t), t), \dot{\bar{x}}_v(t) = F_v^r(\bar{x}_p(t), \bar{x}_v(t), u(t), t) \\ &\quad + \Psi_v(\bar{x}_p(\tau), \bar{x}_v(\tau-), w_{1\tau}(\cdot), \tau) \delta(t - \tau) \\ \bar{\xi}(t) &= \bar{H}(\bar{x}_p(t), \bar{x}_v(t), t) \end{aligned} \quad (21)$$

with  $\bar{x}_p(0) = x_p(0)$ ,  $\bar{x}_v(0) = x_v(0)$ ,  $\bar{x}_p(\tau) = x_p(\tau)$ ,  $\bar{x}_v(\tau-) = x_v(\tau-)$ ;

iii) for any solution  $(\bar{x}_p(t), \bar{x}_v(t), \bar{\xi}(t))$  of the system (21) generated by some admissible controls  $u(t), w_{1\tau}(\cdot)$  there exists a sequence of solutions  $(x_p^\mu(t), x_v^\mu(t), \xi^\mu(t))$  of the system (1), (3) generated by controls

$$\begin{aligned} u^\mu(t) &= \begin{cases} u(t), & t \in [0, \tau], \\ \text{any admissible,} & t > \tau \end{cases} \\ w_1^\mu(\xi, t) &= \begin{cases} w_1(\xi, \sqrt{\mu}(t - \tau)), & t \in [\tau, \tau + t_\mu^*] \\ \text{any admissible,} & \text{otherwise} \end{cases} \end{aligned} \quad (22)$$

where  $t_\mu^* = \min\{s^*(\tau)/\sqrt{\mu}, s_\mu^*(\tau)/\sqrt{\mu}\}$ , which pointwise converge to  $(\bar{x}_p(t), \bar{x}_v(t), \bar{\xi}(t))$  on an interval  $[0, \tau + \varepsilon]$  as  $\mu \rightarrow \infty$ .

The extension of Theorem 1 to the single-impact sequence case can be carried out following Theorems 3 and 6 of [1] and is omitted.

*Singular Phase Description—Equations of Infinitesimal Dynamics:* The modeling objective is to obtain the velocity jump representation corresponding to the limit motion as  $\mu \rightarrow \infty$ . The control objective is to find an impulsive control law which minimizes the velocity of the ball bounce after the impact. Applying transformation (9) of [2] to system (10)–(14), (16) in the new variables  $(y_p, y_v, Y_p, Y_v, \eta)$ , describing the motion in the stretched space-time scale, takes the form

$$\begin{aligned} \dot{y}_p(s) &= y_v(s), \quad \dot{Y}_p(s) = Y_v(s) \\ \dot{y}_v(s) &= -y_p(s) + Y_p(s) - 2\kappa y_v(s) + 2\kappa Y_v(s) \\ \dot{Y}_v(s) &= M^{-1}(w(s) + y_p(s) - Y_p(s) + 2\kappa(y_v(s) - Y_v(s))) \\ \eta(s) &= y_p(s) - Y_p(s) + 2\kappa(y_v(s) - Y_v(s)) \end{aligned} \quad (23)$$

with the initial conditions

$$\begin{aligned} y_p(0) &= Y_p(0) = 0, \quad y_v(0) = x_v(\tau-), \\ Y_v(0) &= X_v(\tau-), \quad \eta(0) = 2\kappa(x_v(\tau-) - X_v(\tau-)). \end{aligned} \quad (24)$$

To address the modeling and the control objectives, introduce the relative coordinates  $q(s) = (q_p(s), q_v(s))$ :  $q_p(s) = y_p(s) - Y_p(s)$ ,  $q_v(s) = y_v(s) - Y_v(s)$ , and let  $h(s) \triangleq M^{-1}w(s)$ . Then, (23) takes the form

$$\begin{aligned} \dot{q}_p(s) &= q_v(s), \quad \dot{q}_v(s) = -h(s) - aq_p(s) - 2a\kappa q_v(s) \\ \dot{y}_v(s) &= -q_p(s) - 2\kappa q_v(s) = -\eta(s) \end{aligned} \quad (25)$$

where  $a = 1 + M^{-1}$ ,  $q_p(0) = 0$ ,  $q_v(0) = x_v(\tau-) - X_v(\tau-) < 0$ , and  $\eta(0) = 2\kappa q_v(0)$ .

*Optimal Control Problem: System Constraints:* Suppose the viscoelastic force is characterized by the so-called restitution (repulsive) property ([1, III-B, p. 48]), i.e. guarantees the repulsion of the ball from the inhibited domain in a finite time without an external force. This means that the rebound conditions

$$q_p(s^*) = 0, \quad \dot{q}_p(s^*) = q_v(s^*) > 0, \quad (26)$$

take place at the instant  $s^*$ —the time moment, in the fast time  $s$ , of the limit system (25) exit from the constraint—given by (17). In this case, system (25) admits an explicit solution

$$q_p(s) = e^{-\lambda s} \left[ q_v(0) \frac{\sin(\omega s)}{\omega} - \int_0^s e^{\lambda s'} \frac{\sin \omega(s - s')}{\omega} h(s') ds' \right] \quad (27)$$

where  $\lambda = \kappa a$ ,  $\omega^2 = a - \lambda^2 > 0$ , and, hence, the restitution condition gives  $a^{-1/2} > \kappa$ .

In new notations the constraints (13), (8) take the form

$$|h(\cdot)| \leq h_0 < \infty, \quad \text{where } h_0 = M^{-1}w_0 \quad (28)$$

$$\int_0^{s^*} |h(s)| ds \leq c, \quad \text{with } c = M^{-1}C. \quad (29)$$

*The Control Objective:* The control objective is to reduce the ball velocity at an instant  $s^*$ . Therefore, it is natural to take as a performance criterion

$$y_v^2(s^*) \rightarrow \min. \quad (30)$$

Without restrictions (7) in the regular motion phase and (29) in the singular one, this problem is solved in [2] with the help of Pontrjagin’s maximum principle. In that case, an optimal control has the form of a single impulse or a single temporal multi-impact depending on the initial conditions and a mass  $M$  of the racket. *Restrictions (7), (29) cardinally change the structure of the optimal control, giving rise to the single-impact or the temporal multi-impact sequences.*

Note, that if the constant  $c$  in (29) is sufficiently large, an optimal control will always satisfy the strict inequality (29), reducing the solution to that of the example in [2]. For this reason, we will consider the case when the constant  $c$  is such that

$$\int_0^{s^*} |h(s)| ds = c < h_0 s^*. \quad (31)$$

*Control Law Synthesis:* To transfer the problem to the form that has no explicit restriction (31) and admits application of Pontrjagin’s maximum principle in its classical form, introduce a new auxiliary variable  $z(s)$  in terms of an equation  $\dot{z}(s) = |h(s)|$ ,  $z(0) = 0$ , with the terminal condition

$$z(s^*) = c. \quad (32)$$

For the optimal control problem (25)–(32), the Hamiltonian  $\mathcal{H} = \mathcal{H}(q_p, q_v, y_v, z, \psi_p, \psi_v, \psi_y, \psi_z, h)$  takes the form

$$\mathcal{H} = \psi_p q_v - \psi_v h - (a\psi_v + \psi_y)(q_p + 2\kappa q_v) + \psi_z |h| \rightarrow \max. \quad (33)$$

The adjoint system is given by

$$\begin{aligned} \dot{\psi}_p(s) &= a\psi_v(s) + \psi_y(s), \quad \dot{\psi}_v(s) = -\psi_p(s) + 2a\kappa\psi_v(s) \\ &\quad + 2\kappa\psi_y(s), \quad \dot{\psi}_y(s) = 0, \text{ i.e. } \psi_y = C_y = \text{const}, \\ \dot{\psi}_z(s) &= 0, \text{ i.e. } \psi_z = C_z = \text{const}. \end{aligned} \quad (34)$$

The terminal transversality conditions at  $s = s^*$  take the form

$$2y_v \delta y_v + \psi_p \delta q_p + \psi_v \delta q_v + \psi_y \delta y_v + \psi_z \delta z - \mathcal{H} \delta s = 0 \quad (35)$$

where  $\delta q_p = 0$  and  $\delta z = 0$  due to conditions (26) and (32), respectively. This yields

$$2y_v(s^*) + C_y = 0, \quad \psi_v(s^*) = 0, \quad \mathcal{H}(s^*) = 0. \quad (36)$$

Then,  $\mathcal{H}(s^*) = 0$  implies

$$\psi_p(s^*) - 2\kappa C_y = 0. \quad (37)$$

Relations (36), (37) give the terminal conditions for (34) of adjoint variables  $\psi_v(s)$ ,  $\psi_p(s)$  that determine the optimal control signal and may be easily integrated backwards in time.

The application of the Pontrjagin's technique to the system analogous to (34), (35) is considered in detail in [2]. The main difference between the present case and that of [2] is that in [2] an optimal control  $\tilde{h}$  maximizes a linear function  $f(h) = -\psi_v h$ , so that  $\tilde{h}(s) = -h_0 \text{sign}(\psi_v(s))$ , whereas in the present case an optimal control  $\tilde{u}$  maximizes a piecewise-linear function

$$f(h) = kh - |h| \rightarrow \max \quad (38)$$

where  $k = k(s) = \psi_v(s)/C_z$ , with  $C_z < 0$ . It then follows from (38) that

$$\tilde{h}(s) = \begin{cases} h_0, & k(s) > 1 \\ 0, & k(s) \in [-1, 1] \\ -h_0, & k(s) < -1. \end{cases} \quad (39)$$

It should be noted that for some initial conditions it is possible not just to minimize a rebound velocity but to bring the ball to a full stop. In the latter case, the optimal control problem stated above becomes degenerate due to the appearance of an additional terminal condition  $y_v(s^*) = 0$ . Indeed, as it follows from (36), (37), this condition gives  $C_y = \psi_v(s^*) = 0$  and  $\psi_p(s^*) = 0$ , leading to the trivial solution of the adjoint system (34). Considering therefore an alternative criterion, it is natural to take minimization of the terminal time:

$$\int_0^{s^*} C_h ds \rightarrow \min, \quad C_h > 0 \text{ any constant.} \quad (40)$$

Hamiltonian, maximum principle, and an optimal control law retain the same form (33), (39), (34) for the new time-optimal control problem (25), (26),  $y_v(s^*) = 0$ , (40). But due to  $y_v(s^*) = 0$  and (40), the transversality conditions take the form  $\psi_v(s^*) = 0$ ,  $\mathcal{H}(s^*) = C_h$ . As above, these conditions permit obtaining the boundary conditions for integrating the system (34) backwards in time and calculating the control mode switching function  $k(s)$  in (39).

The constraint (8) is shown below to give rise to the single-impulse sequence for the sufficiently small  $C_i$ , irrespectively of the value of  $T_k$ . This sequence turns into the temporal multi-impact sequence when the size of the constraint is increased. When the constraint exceeds certain bound, essentially rendering the system unconstrained, the sequence could be shown to reduce to a single temporal multi-impact case given in [2].

*Unconstrained External Mechanical Impulse:* In system (23)–(25) with the parameters and the initial data as in the example in [2] ( $a = 2$ ,  $\kappa = 0.25$ ,  $h_0 = 0.3$ ,  $q_p(0) = 0$ ,  $q_v(0) = -0.0822$ ,  $y_v(0) = -0.3329$ ,  $Y_v(0) = -0.2507$ ) the ball could be stopped by a single double-impact control with impulses  $p_1 = w_0 \tilde{s}$  and  $p_2 = -w_0(s^* - \tilde{s})$ , where  $\tilde{s} = s_1 = 1.871$  and  $s^* = s_2 = 2.835$ .

*Constrained External Mechanical Impulse:* Now, introduce constraints (8) with  $C = 0.3$  (7). Under the aforementioned initial conditions of the example in [2] the latter implies that in (31)  $c = 0.3$  as well. In this case, the maximum permissible external impulse  $p =$

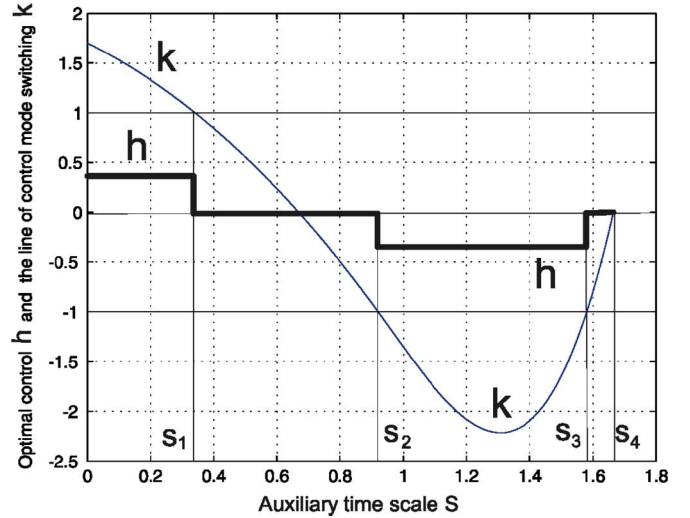


Fig. 2. Control mode switching function  $k$  and the piece-wise constant optimal control  $h$  that stops the ball.

$w_0 s_1 = M h_0 s_1 = 0.3$  corresponding to the step-wise optimal control law

$$\tilde{h}(s) = \begin{cases} h_0, & s < s_1 \\ 0, & s \geq s_1 \end{cases} \quad (41)$$

where  $s_1 = 1$ , does not stop the ball. The racket in this case is moving away from the ball with the relative velocity  $q_v(s^*) = 0.10414$ . Here the terminal instant  $s^* = s_2 = 2.823$  and the terminal values of the absolute velocities of the ball and the racket are equal to  $y_v(s^*) = -0.089753$  and  $Y_v(s^*) = -0.1939$ , respectively. The step  $h_0$ ,  $s < s_1$ , of the optimal control (41) is located at the beginning of the control time interval  $[0, s_2]$ . Such location maximizes the work of the external force on a ball/racket system and hence minimizes the absolute velocity of the ball. For finite  $\mu$  transformation (9) of [2] leads to the system (25) directly, yielding the exact optimal solution.

*1) Ball/Racket System Under a Single-Impact Sequence:* Let in (6)  $F_0 = 0.2$  and in (7)  $T_k = T = 1$  for any  $k$ . Then, after time  $T$  the racket undergoes impulsive action  $F_0 = 0.2$  that generates a jump of the racket velocity  $\Delta Y_v = 0.2$ . Therefore, new racket velocity takes the value  $Y_v = 0.0061$  in a positive direction. Relative closing speed between the ball and the racket becomes  $q_v = y_v - Y_v = -0.09585$ . After the second collision the ball could be stopped with the help of a single double-impulse control with impulses  $p_1 = w_0 s_1$  and  $p_2 = -w_0(s_3 - s_2)$ , where  $s_1 = 0.34$ ,  $s_2 = 0.92$ ,  $s_3 = 1.54$ . In the singular phase, the piece-wise constant control law corresponding to these impulses is depicted in Fig. 2. Due to the profile of the control mode switching function  $k(s)$ , optimal control  $\tilde{h}(s)$  undergoes jumps at instants  $s_1, s_2, s_3$ . Stopping time is calculated to be  $s^* = s_4 = 1.664$ .

*Remark 1:* This dependence can be applied as a function of pressure measurable by a built into the racket pressure sensor.

*2) Temporal Multi-Impact Sequences:* In the case of the single temporal multi-impact,  $F_v^{rs}(x_p, x_v, w_2^\mu(\xi, t), t, \mu) \neq 0$  and the infinitesimal dynamics equation is given by (41) of [2]. This admits natural generalization of the single temporal multi-impact Theorems 2 and 3 of [2] to the temporal multi-impact sequences given next.

*Limit System Representation and Control Law Implementation Under a Temporal Multi-Impact Sequence:* The sequence  $\{\tau_i\}$ ,  $i = 1, \dots, N$ , is defined as that in (34) of [1]. But the double sequence  $\{\tau_i^\mu, \tau_i^{\mu*}\}$ ,  $i = 1, \dots, N$ , splits into  $N$  finite series

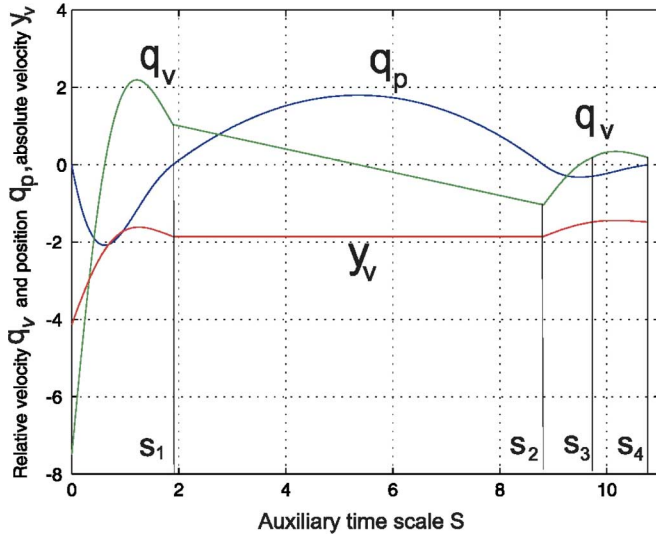


Fig. 3. Variation of  $q_p$ ,  $q_v$  and  $y_v$  in the inhibited domain in the case of the first singular phase under constant control  $h = h_0 = 0.3$ .

$\tau_i^\mu = \tau_{i1}^\mu < \tau_{i1}^{\mu*} < \tau_{i2}^\mu < \tau_{i2}^{\mu*} < \dots < \tau_{iN_i}^\mu < \tau_{iN_i}^{\mu*} = \tau_i^{\mu*}$ , where  $\tau_{ij}^\mu = \tau_i^\mu + \mu^{-1/2} s_{\mu j}(\tau_i^\mu)$ ,  $\tau_{ij}^{\mu*} = \tau_i^{\mu*} + \mu^{-1/2} s_{\mu j}^*(\tau_i^{\mu*})$ ,  $j = 1, \dots, N_i$ . Here each series corresponds to one temporal multi-impact.

**Theorem 2:** Let  $(\bar{x}_p(t), \bar{x}_v(t)), t \in [0, T]$ , be a solution of the system (39) of [1] with shift operators replaced by  $\Psi_v(y_p(0), y_v(0), w_{1\tau}(\cdot), w_{2\tau}(\cdot), \tau)$  under some admissible controls  $u, w_1$ , and  $w_2$ . Then, if  $\mu \rightarrow \infty$ , the corresponding sequence of ordinary solutions  $(x_p^\mu(t), x_v^\mu(t))$  of the system (1) with the same control signals  $u(t), t \in [0, T]$ , and  $\gamma^\mu(\xi, t) = \gamma(\xi, \sqrt{\mu}(t - \tau_i^\mu))$ ,  $t \in [\tau_i^\mu, \tau_i^{\mu*}]$ , converges everywhere on  $[0, T]$ , except, possibly, at the points  $\{\tau_i\}$ , to the general solution  $(\bar{x}_p(t), \bar{x}_v(t))$  of the modified system (39) of [1]. Here  $\gamma$  is any of the controls  $w_1$  and  $w_2$  which, generally, admit an extension in the neighborhood of the points  $\tau_i^{\mu*}$ .

**Ball/Racket System Under Constraint on Impulse. Constrained Temporal Multi-Impact:** Now consider the case of a light racket ( $M = 1/3$  and hence  $a = 1 + 1/M = 4$ ), which moves towards the ball with a higher initial relative velocity  $q_v(0) = -7.5$ . The other parameters and initial conditions of the model are  $\kappa = 0.25$ ,  $h_0 = 0.3$ ,  $q_p(0) = 0$ ,  $y_v(0) = -4.1322$ ,  $Y_v(0) = 3.3678$ ,  $C = 0.9725$ , hence  $c = C/M = 2.9175$ .  $C$  is now seen to be more than three times greater than that in the previous example. Then, considering motion on  $0 < s_1 < s_2 < s_3 < s_4 < \infty$ . Fig. 3 illustrates this case. In the interval  $[0, s_1]$  the external control action is  $h(s) = h_0 = 0.3$ . In this interval the ball moves inside the racket. At time  $s_1$  the ball and the racket disengage and during the time interval  $[s_1, s_2]$  they move separately: the ball—with a constant speed  $y(s) = y(s_1) = \text{const}$  in the constraint-free domain and the racket—with a linear increasing speed  $Y(s) = y(s_1) - q(s)$  under a constant external control action  $h(s) = h_0 = 0.3$ . The latter action generates the force applied to the racket *outside of the contact phase*. To distinguish this force from the ones acting in the singular, or contact, phases of the optimal force sequence, it will be further denoted as  $F^{rs}$ , and the motion phase corresponding to it will be referred to as *the intersingular*. At time  $s_2$  the racket collides with the ball once again. During the time interval  $[s_2, s_4]$  the ball moves inside the racket. At time  $s_3$  the optimal control equals zero due to impulse depletion—manifestation of the constraint (8) on the admissible impulse depending on the constant  $C$ . Finally, at time  $s_4$  the ball and the racket are moving away, with nonzero ball velocity.

The force  $F^{rs}$  is a part of the optimal force sequence, further referred to as *the temporal multi-impact*, that induces contact and non-contact phases on the time interval cumulatively going to zero as  $\mu \rightarrow \infty$ . Since the corresponding motion is characterized by contact interruption, yet its total duration tends to zero as  $\mu \rightarrow \infty$ , it is further referred to as *the interlaced singular phase*.

**Temporal Multi-Impact Sequence:** The initial absolute velocity of the ball exiting the first temporal multi-impact is  $y_v = -1.4802$ . Applying after time  $T_k = T = 1$  the impulsive action  $F_0 = 1.14$  generates a jump of the racket velocity  $\Delta Y_v = 3.4177$ . Therefore, the new racket velocity takes the value  $Y_v = 1.7493$  in the positive direction and the relative closing speed between the ball and the racket becomes  $q_v = y_v - Y_v = -3.2302$ . Now, as shown in ([2], p. 1745), the ball could be stopped by a single temporal multi-impact control with impulses  $p_1 = w_0 s_3 = 0.90766$  and  $p_2 = -w_0(s_4 - s_3) = -0.06484$ , where  $s_3 = 9.0766$ ,  $s_4 = s_2^* = 9.725$ , and  $w_0 = M u_0 = 0.1$ , without violating constraint (8).

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